Lois usuelles - Espérance / Variance

Escoffier

I Loi uniforme.

$X \sim U(\llbracket 1,n \rrbracket)$ si $X(\Omega) = \llbracket 1,n \rrbracket$

$\forall i \in \llbracket 1,n \rrbracket \ \mathbb{P}(X=i) = \frac{1}{n}$

$$\mathbb{E}(X) = \frac{n+1}{2} \quad \text{et} \quad \mathbb{V}(X) = \frac{n^2-1}{12}$$

preuve $\mathbb{E}(X) = \sum_{i=1}^{n} i \times \frac{1}{n} = \frac{1}{n} \times \frac{n(n+1)}{2} = \frac{n+1}{2}$

$\mathbb{E}(X^2) = \sum_{i=1}^{n} i^2 \times \frac{1}{n} = \frac{1}{n} \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$

$\mathbb{V}(X) = E(X^2) - (E(X))^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{(n+1)(2(2n+1)-3(n+1))}{12}$

$= \frac{(n+1)(4n-2)}{12} = \frac{(n+1)(n-1)}{12}$

II Bernoulli

$p \in [0,1]$, $X \sim \mathcal{B}(p)$ $\begin{cases} \mathbb{P}(X=0) = 1-p \\ \mathbb{P}(X=1) = p \end{cases}$

$$\mathbb{E}(X) = p \qquad \mathbb{V}(X) = p(1-p)$$

preuve : $\mathbb{E}(X) = 0 \times \mathbb{P}(X=0) + 1\,\mathbb{P}(X=1) = p$

$E(X^2) = 0 \times \mathbb{P}(X=0) + \mathbb{P}(X=1) = p$

$\mathbb{V}(X) = E(X^2) - E(X)^2 = p - p^2 = p(1-p)$

III Binomiale

$n \in \mathbb{N}^*$ $p \in [0,1]$ $X \sim \mathcal{B}(n,p)$, $X(\Omega) = \llbracket 0,n \rrbracket$

$\forall k \in \llbracket 0,n \rrbracket \ \mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$\mathbb{E}(X) = np$ et $\mathbb{V}(X) = np(1-p)$

$$\mathbb{E}(X) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} n \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)} \quad \text{(Binôme)}$$

$$= np$$

Pour la variance $k^2 = k(k-1) + k$.

IV Loi de Poisson.

$\lambda > 0$, $X \sim \mathcal{P}(\lambda)$ si $X(\Omega) = \mathbb{N}$ et $\forall k \in \mathbb{N}$ $\mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$

$\Rightarrow$ loi limite d'1 binomiale $\mathcal{B}(n, \frac{\lambda}{n})$ avec $\lambda = np$

$\mathbb{E}(X) = \lambda$ et $\mathbb{V}(X) = \lambda$

$$\mathbb{E}(X) = \sum_{k=0}^{+\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{+\infty} \frac{\lambda^k}{(k-1)!} = e^{-\lambda} \lambda e^{\lambda} = \lambda \quad \text{(cvg par d'Alembert)}$$

$$\mathbb{E}(X^2) = \sum_{k=0}^{+\infty} k^2 e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=0}^{+\infty} (k(k-1) + k) e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \sum_{k=0}^{+\infty} k(k-1) e^{-\lambda} \frac{\lambda^k}{k!} + \underbrace{\sum_{k=0}^{+\infty} k e^{-\lambda} \frac{\lambda^k}{k!}}_{=\lambda}$$

$$= e^{-\lambda} \sum_{k=2}^{+\infty} \frac{\lambda^k}{(k-2)!} + \lambda$$

$$= e^{-\lambda} \lambda^2 e^{\lambda} + \lambda = \lambda^2 + \lambda$$

et $\mathbb{V}(X) = E(X^2) - \mathbb{E}(X)^2 = \lambda$.